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Ring-shaped quasisoliton solutions of the circular symmetric 2D sine-Gordon equation

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Abstract. In this paper, the circular symmetric two-dimensional sine-Gordon equation omitting the origin is investigated by using the numerical integral method. There exists a ring-shaped quasisoliton solution and this ring-shaped wave firstly travels outward and then at certain positions it returns. A simple analytical treatment of return time is made and is in agreement with the numerical one. Finally, an interpretation of the return effect is also presented.

In many fields of physics, one-dimensional solitary wave solutions have been investigated by many people both theoretically and experimentally [1]. A explicit analytical lumped soliton solution of the two-dimensional Kadomtsev–Petviashvili equation has been found by Manakov *et al* [2] and Ablowitz and Satsuma [3]. Spherically and cylindrically symmetric solitary wave solutions of modified Korteweg–de Vries equations were derived by Maxon and Viecelli [4]. Their numerical and analytical investigations show that these waves differ qualitatively from the one-dimensional solution. In contrast to the one-dimensional case, where the wave form is retained, these solutions increase in amplitude and decrease in width as they shrink. The existence of two-dimensional solitary waves was experimentally verified by Herschkowitz and Romesser [5].

Christiansen and Olsen have numerically investigated symmetric solitary wave solutions of the sine-Gordon equation in two and three spatial dimensions [6]. They found that there exists a ring-shaped quasisoliton solution. The ring-shaped wave firstly travels outward and then at certain position it returns. This motion of the ring wave is called the return effect. But for the geometric dimension they included the origin. In our opinion the origin seems to cause some trouble [6] (see equation (1′)). So in the present paper we want to discuss the quasisoliton solution of an annular sine-Gordon equation with circular symmetry. On the other hand, in physics, the circular symmetric two-dimensional Josephson junction is actually described by this circular symmetric two-dimensional sine-Gordon equation [7]. In [7], we have discussed the static solution of this equation. As the first step for discussing the dynamical behaviour further, we discuss the quasisoliton solution here.

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For the annular Josephson junction with the dissipative term omitted, we have the circular symmetric sine-Gordon equation:

$$\Phi_{rr} + \Phi_r / r - \Phi_{tt} = \sin \Phi \tag{1}$$

where $\Phi(r, t)$ is the phase of the superconducting order parameter, the internal radius is r_I and the external radius is r_E. In all of our computations, r_I is equal to 20. We compute this equation by using finite-difference methods for spatial dimension with the boundary conditions

$$\Phi(r = r_I, t = 0) = 0 \tag{2}$$

$$\Phi_t(r = r_I, t = 0) = 0 \tag{3}$$

and assuming the ring-shaped soliton solution at time $t = 0$ is

$$\Phi(r, 0) = 4 \tan^{-1} \exp \frac{r - r_0}{(1 - u^2)^{1/2}} \tag{4}$$

$$\Phi_t(r, 0) = \frac{-2u}{(1 - u^2)^{1/2}} \operatorname{sech} \frac{r - r_0}{(1 - u^2)^{1/2}}. \tag{5}$$

Here r_0 is the initial position of the ring wave and u is the outward initial expansion velocity. The results for the radius derivative of Φ, Φ_r, for $r_0 = 30$ and $u = 0.8$ are shown in figure 1. It is clear that for $t = t_{ret}$ the ring wave attains its maximum radius r_{max}; then the ring wave returns. The return effect occurs and the ring wave shrinks forward its internal boundary r_I.

After the reflection the expanding ring wave splits into a ring wave with smaller amplitude and a ring wave with larger amplitude. The smaller wave radiates into infinity while the larger one experiences a return effect. The resulting shrinking wave is then reflected at $r = r_I$ again. The radiated phenomenon and the return effect described above are repeated several times.

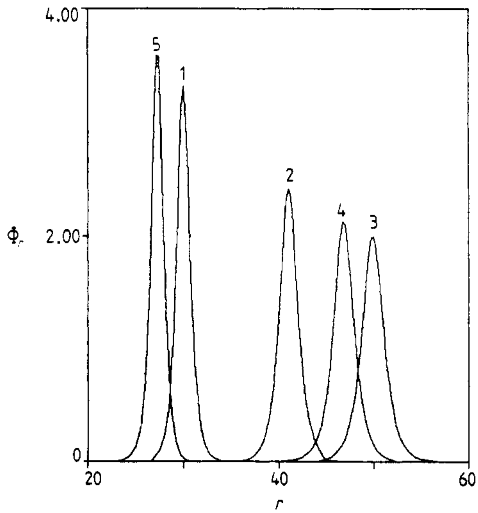

Figure 1. Evolution with time of the numerical solution of the initial-value problem of (1)–(5). The initial position of the ring wave is $r_0 = 30$ and the initial outward expansion velocity is $u = 0.8$. The vertical axis is the radius derivative of Φ, Φ_r. The peaks are for: (1) $t = 0$, (2) $t = 16$, (3) $t = 48$, (4) $t = 68$, (5) $t = 96$. They denote the expanding ring wave before returning (1, 2); just at the returning point (3) and the shrinking ring wave after returning (4, 5).

An analytical approximation for the ring wave returning are derived in the following. From figure 1 we can see that the shape of the ring waves are not changed drastically as time passes [6]. So we introduced the assumption

$$\Phi(r, t) = 4 \tan^{-1} \exp\left(\frac{r - R(t)}{(1 - R'(t)^2)^{1/2}}\right). \tag{6}$$

Here the function $R(t)$ describes the position of the maximum of Φ_r.

From (1)-(3), we get

$$\frac{1}{R} = -R''(t) \frac{1 + R'(t)^2}{1 - R'(t)^2} \tag{7}$$

$$R(t = 0) = r_0 \tag{8}$$

$$R'(t = 0) = u. \tag{9}$$

A first integral of the above equations is

$$\frac{R}{r_0} = \frac{1 - R'(t)^2}{1 - u^2} \exp\left(\frac{R'(t)^2 - u^2}{2}\right). \tag{10}$$

If R reaches the maximum, $R = r_{\max}$, and then R' is equal to zero at the same time; $r_{\max}$ is thus

$$r_{\max} = r_0 \frac{1}{1 - u^2} \exp\left(\frac{-u^2}{2}\right). \tag{11}$$

From (10) and (11), we can obtain t_{ret} as

$$t_{\text{ret}} = \frac{2^{1/2} r_0}{(1 - u^2) \exp(u^2/2)} [1 - (1 - u^2) \exp(u^2/2)]^{1/2}. \tag{12}$$

We show the theoretical and numerical results in figure 2. We can see that there is very good agreement for small u.

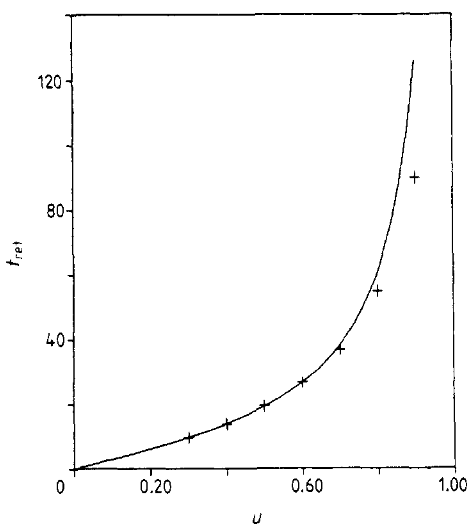

Figure 2. Return time plotted against initial velocity of ring wave. The curve is the result of (12) and the points are the results of numerical calculations for ring waves with an initial position $r_0 = 30$.

In figure 3(a, b), we show the interaction between two ring waves:

$$\Phi(r,0)=4\tan^{-1}\exp\left(\frac{r-r_1}{(1-u_1^2)^{1/2}}\right)+4\tan^{-1}\exp\left(\frac{\pm(r-r_2)}{(1-u_2^2)^{1/2}}\right) \tag{13}$$

$$\Phi_t(r,0)=\frac{-2u_1}{(1-u_1^2)^{1/2}}\operatorname{sech}\left(\frac{r-r_1}{1-u_1^2}\right)\pm\frac{2u_2}{(1-u_1^2)^{1/2}}\operatorname{sech}\left(\frac{r-r_2}{(1-u_2^2)^{1/2}}\right). \tag{14}$$

Here the upper signs result in a quasisoliton-quasisoliton collision while the lower ones result in a quasisoliton-antiquasisoliton collision. In figure 3, it is seen that the two waves collide.

The return effect may be interpreted as follows. Since the width and the velocity of the ring wave can be derived from (10)

$$W=(1-R'(t)^2)^{1/2}=\frac{1}{(r_{\max})^{1/2}}(2R(t)-r_{\max})^{1/2} \tag{15}$$

$$R'=\left(\frac{2}{r_{\max}}\right)^{1/2}(r_{\max}-R(t))^{1/2}. \tag{16}$$

From (15) and (16), we can see that as the wave is travelling outward, the velocity, R', of the wave slows down and the width, W, of the wave increases. As the wave is travelling to $r=r_{\max}$ the velocity slows down to zero and the width of the ring wave is increased to its maximum. Because the ring wave is unstable at $r_{\max}$, it must return. Since there is a boundary at $r=r_1$, the wave must move backward and forward between r_1 and $r_{\max}$.

In conclusion we have numerically studied the circular symmetric two-dimensional sine-Gordon equation. In contrast to the previous work, we omitted the origin, the natural singular point, of (1) in our case. Our results show that there exists a ring-shaped quasisoliton solution and this ring-shaped wave firstly travels outward and then at certain position it returns. There also exists a collision between the quasisolition and soliton. From the assumption of no drastic change of the ring wave with time, a simple

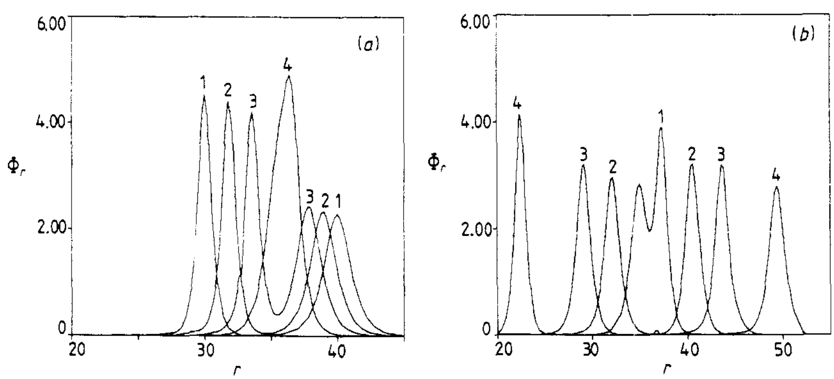

Figure 3. (a) Evolution with time of the numerical solution of the initial-value problem of (1)–(3) and (13) and (14) with upper signs in (13) and (14). The initial position of one ring wave is $r_1=30$ and that of another is $r_2=40$, and the correspondent initial velocities are $u_1=0.9$ (outward) and $u_2=0.5$ (inward). The vertical axis is the radius derivative of Φ, Φ_r. The peaks are marked: (1) $t=0$, (2) $t=2$, (3) $t=4$, (4) $t=7$, denoting the ring waves before collision (1, 2, 3) and just at collision (4). (b) Continuation of the solution in (a) for (1) $t=8$, (2) $t=12$, (3) $t=16$, (4) $t=24$, denoting the ring waves after collision.

analytical treatment of return time is made and is in agreement with the numerical one. An interpretation of the return effect is also presented. However, we want to point out that the r_{max} becomes smaller and smaller with a slow speed since there is little energy lost as the wave is moving (some small-amplitude ring waves that result from the reflection of the ring wave at r_l radiate into infinity).

In addition, for the circular symmetric two-dimensional sine-Gordon equation, the more interesting problem is to study the solutions of the equation within an annular region. This is the dynamical problem of the annular Josephson junction, and it seems that there are no stable ring-shaped quasisoliton solutions [8], which might well deserve further study.

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